# Exercise 17

Solve the initial-value problem.

$$y'' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 3$$

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### Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Plug these formulas into the ODE.

$$r^2e^{rx} + 3(e^{rx}) = 0$$

Divide both sides by $e^{rx}$.

$$r^2 + 3 = 0$$

Solve for $r$.

$$r = \left\{-i\sqrt{3}, i\sqrt{3}\right\}$$

Two solutions to the ODE are $e^{-i\sqrt{3}x}$ and $e^{i\sqrt{3}x}$. By the principle of superposition, then,

$$\begin{aligned}
y(x) &= C_1e^{-i\sqrt{3}x} + C_2e^{i\sqrt{3}x} \\
&= C_1(\cos\sqrt{3}x - i\sin\sqrt{3}x) + C_2(\cos\sqrt{3}x + i\sin\sqrt{3}x) \\
&= (C_1 + C_2)\cos\sqrt{3}x + (-iC_1 + iC_2)\sin\sqrt{3}x \\
&= C_3\cos\sqrt{3}x + C_4\sin\sqrt{3}x,
\end{aligned}$$

where $C_1$, $C_2$, $C_3$, and $C_4$ are arbitrary constants. Differentiate this general solution.

$$y'(x) = -\sqrt{3}C_3\sin\sqrt{3}x + \sqrt{3}C_4\cos\sqrt{3}x$$

Apply the initial conditions to determine $C_3$ and $C_4$.

$$y(0) = C_3 = 1$$

$$y'(0) = \sqrt{3}C_4 = 3$$

Solving this system of equations yields $C_3 = 1$ and $C_4 = \sqrt{3}$. Therefore, the solution to the initial value problem is

$$y(x) = \cos\sqrt{3}x + \sqrt{3}\sin\sqrt{3}x.$$

Below is a graph of $y(x)$ versus $x$.